On Exactness Of The Supersymmetric WKB Approximation Scheme

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Abstract

Exactness of the lowest order supersymmetric WKB (SWKB) quantization condition $\int_{x_1}^{x_2} \sqrt{E-\omega^2(x)}dx = n\hbar\pi$, for certain potentials, is examined, using complex integration technique. Comparison of the above scheme with a similar, but *exact* quantization condition, $\oint_c p(x,E)dx = 2\pi n\hbar$, originating from the quantum Hamilton-Jacobi formalism reveals that, the locations and the residues of the poles that contribute to these integrals match identically, for both of these cases. As these poles completely determine the eigenvalues in these two cases, the exactness of the SWKB for these potentials is accounted for. Three non-exact cases are also analysed; the origin of this non-exactness is shown to be due the presence of additional singularities in $\sqrt{E-\omega^2(x)}$, like branch cuts in the $x-$plane.

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Exactness of the lowest order supersymmetric WKB (SWKB) approximation scheme for a class of potentials has been the subject of wide discussions in the literature [1]. Explicit calculations, akin to that of the celebrated harmonic oscillator case [2], first revealed the absence of $O(\hbar^2)$ to $O(\hbar^6)$ corrections to the lowest order result [3]. Subsequently, for one of the above potentials, it was shown that all higher order corrections vanish identically [4]. A discrete reparameterisation symmetry, the so-called shape invariance, of these potentials, has been thought to be responsible for this exactness [5]. Recently, a new class of shape invariant potentials have been found, for which the exactness of SWKB quantization does not hold true [6]. In this light and considering the importance of this quantization scheme in the potential problems, a deeper analysis of the origin of this exactness is of utmost significance.

In supersymmetric (SUSY) quantum mechanics [7], a pair of Hamiltonians, $H_\pm$ with potentials $V_\pm(x)$, given by $\omega^2(x) \pm \hbar\partial\omega(x)/\partial x$, $(2m = 1)$, have identical bound state spectra, except for the ground state; $\omega(x)$ here, is the superpotential in SUSY quantum mechanics. In the case of unbroken SUSY, conventionally, the energy of the ground state of H_-, which is unpaired, is taken to be zero. It was observed in [1], that a semi-classical WKB type approximation $\int_{x_1}^{x_2} \sqrt{E - \omega^2(x)}dx = n\hbar\pi$, making use of $(V_-(x) + V_+(x))/2 = \omega^2(x)$, gave *exact* energy eigenvalues, for many well known potentials. This is to be contrasted with the conventional WKB scheme, which reproduced the exact spectra only for the harmonic oscillator and the Morse potentials.

In what follows, we will first use a complex integration technique, to evaluate the SWKB integral given by,

$$J_{SWKB} \equiv \frac{1}{\pi}\int_{x_1}^{x_2} \sqrt{E - \omega^2(x)}dx = \frac{1}{2\pi}\oint_C \sqrt{E - \omega^2(x)}dx = n\hbar \quad . \tag{1}$$

Here, x_1 and x_2 $(x_1 < x_2)$ are the turning points, which are the real roots of $E - \omega^2(x) = 0$ and C is a counter clockwise contour, enclosing the branch cut between x_1 and x_2. Note

that the integrand is a double valued function, which is taken to be positive just below the cut, joining the two turning points [8]. The above integral will, then, be contrasted with an exact quantization condition,

$$J(E) \equiv \frac{1}{2\pi} \oint_{C'} p(x,E)dx = n\hbar \quad , \tag{2}$$

originating from the quantum Hamilton-Jacobi (H-J) formalism [9,10]. Here, $J(E)$ is the quantum action variable and $p(x,E)$ is the quantum momentum function (QMF) and C' is a counter clockwise contour in the complex $x-$plane, enclosing the real line between the classical tuning points. The classical turning points are the real values of x, for which $E - V(x) \equiv p_c^2(x,E)$ vanishes, and encloses the region between which the classical motion takes place. In a previous paper [11] we used the quantization condition given by Eq. (2) to obtain the energy eigenvalues for a set of SUSY potentials. This was done by rewriting the integral $J(E)$ in Eq. (2) as sum of integrals over contours enclosing the singular points of QMF.

In this paper we evaluate J_{SWKB} by complex integration technique in a manner similar to that used in [11] for $J(E)$. Interestingly, we find that the locations and the residues of the poles of QMF, which determine the eigenspectra for the exact quantization scheme match identically with the poles appearing in the evaluation of the integral in Eq. (1). This, then, accounts for the SWKB scheme being exact in these cases.

In this paper we also analyse three potentials for which the energy eigenvalues are known, but SWKB fails to give the exact answer [5,12]; It is seen that the presence of additional singularities, like branch cuts, prevents an exact calculation of SWKB integrals for all these cases. Intriguingly, in the above examples, it is found that, if one ignores the contributions of some of the branch cuts in the non-classical region, the exact answer is reproduced. Hence, for these potentials, the difference between the exact answer and the SWKB approximation scheme, is quantified; this may be quite useful in developing new approximation schemes for

non-exactly solvable problems.

The QMF satisfies the quantum H-J equation ($2m = 1$),

$$\begin{aligned} p^2(x,E) + \frac{\hbar}{i}\frac{\partial p(x,E)}{\partial x} &= E - V(x) \\ &\equiv p_c^2(x,E) \quad . \end{aligned} \tag{3}$$

Here, $V(x)$ is any given potential and $p_c(x,E)$ is the classical momentum function, defined so as to have the value $+\sqrt{E - V(x)}$ just below the branch cut which joins the two turning points. QMF $p(x,E)$ obeys

$$\lim_{\hbar \to 0} p(x,E) \to p_c(x,E) \quad . \tag{4}$$

This can be thought of as a manifestation of the *correspondence principle* or as a *boundary condition* on the QMF. The quantum H-J equation is equivalent to the Schrödinger equation, under the substitution,

$$p(x,E) = \frac{\hbar}{i}\frac{1}{\psi(x,E)}\frac{\partial \psi(x,E)}{\partial x} \quad . \tag{5}$$

The nodes of the wave function $\psi(x,E)$ are known to appear between the classical turning points. QMF $p(x,E)$ will then have poles at the corresponding points, hence $J(E)$ which counts this number, will obey the exact quantization condition, given in Eq. (2). If we substitute an $\hbar$ expansion for $p(x,E)$, the quantization condition (2) coincides with the well known Dunham's formula.

We explicitly workout the SWKB integral in Eq. (1) for a set of trigonometric and hyperbolic potentials and tabulate the results in Table I. In what follows, the steps required to solve one of these potentials are outlined. We then compare these results with our previous calculation [11] using the quantum action variable.

We shall now evaluate J_{SWKB} for the Eckart potential,

$$V_{-}(x) = A^2 + \frac{B^2}{A^2} + A(A - \alpha\hbar)\text{cosech}^2\alpha x - 2B\coth\alpha x \quad . \tag{6}$$

whose superpotential is,

$$\omega(x) = -A \coth \alpha x + B/A \quad , \qquad B > A^2 \text{ and } 0 < x < \infty \quad . \tag{7}$$

We use the mapping $y = \exp(\alpha x)$ in which case,

$$V(y) = A^2 + \frac{B^2}{A^2} + \frac{4A(A - \alpha\hbar)y^2}{(y^2 - 1)^2} - \frac{2B(y^2 + 1)}{(y^2 - 1)} \quad , \tag{8}$$

and

$$\omega(y) = -A\frac{y^2 + 1}{y^2 - 1} + \frac{B}{A} \quad . \tag{9}$$

The quantization condition in Eq. (1) for the SWKB approximation becomes,

$$J_{SWKB} = \frac{1}{2\pi\alpha} \oint_{C_1} \frac{\sqrt{E - \omega^2(y)}}{y} dy = n\hbar \quad . \tag{10}$$

It is clear from Eq. (10) that, the integrand has singularities at $y = 0$ and $y = \pm 1$. The roots of $E - \omega^2(y) = 0$, give the branch points; they are four in number, two of which lie in the classical and the other two in the non-classical ($y < 0$) region of the complex $y-$plane. To evaluate the above integral, consider a counter clockwise contour integral J_{Γ_R} for a circle Γ_R of radius R, which is such that it encloses all the singular points of the above integrand (see Fig. 1). Then,

$$J_{\Gamma_R} = J_{SWKB} + J_{C_2} + J_{\gamma_1} + J_{\gamma_2} + J_{\gamma_3} \quad . \tag{11}$$

Here, J_{C_2} is the integral along the counter clockwise contour C_2 enclosing the branch cut in the Re$y < 0$ region. $J_{\gamma_1}, J_{\gamma_2}$ and J_{γ_3} are the integrals along contours γ_1, γ_2 and γ_3 enclosing the singular points at $y = 1$, $y = -1$ and $y = 0$ respectively. It may be noticed that the symmetry $y \to -y$ of $E - \omega^2(y)$ implies,

$$J_{SWKB} = J_{C_2} \quad . \tag{12}$$

For the pole at $y = 0$, one gets,

$$J_{\gamma_1} = \frac{i\sqrt{E - A^2 - B^2/A^2 - 2B}}{\alpha} \quad . \tag{13}$$

Similarly for poles at $y = \pm 1$,

$$J_{\gamma_2} = J_{\gamma_3} = \frac{A}{\alpha} \quad . \tag{14}$$

Now for the calculation of J_{Γ_R} one more change of variable $z = 1/y$ is sought; so the singularity at $y \to \infty$ is mapped to the singularity at $z = 0$. The contour integral for the pole at $z = 0$ is,

$$J_{\Gamma_R} = \frac{-i\sqrt{E - A^2 - B^2/A^2 + 2B}}{\alpha} \quad . \tag{15}$$

The SWKB quantization condition

$$J_{\Gamma_R} - \sum_{p=1}^{3} J_{\gamma_p} = 2n\hbar \quad , \tag{16}$$

when inverted for E gives

$$E_n = A^2 + B^2/A^2 - \frac{B^2}{(n\alpha\hbar + A)^2} - (n\alpha\hbar + A)^2 \quad . \tag{17}$$

The calculation of eigenvalues for other SUSY potentials proceeds along similar lines and the results are summarised in Table I.

Comparing the above computation of J_{SWKB}, with that of $J(E)$ in our previous paper, we find that the singularity structure of $\sqrt{E - \omega^2}$ other than the branch cuts, match exactly with that of the fixed poles of $p(x, E)$ in the quantum H-J formalism [11]. In all the cases tabulated in Table I, the location of the fixed poles and the values of the corresponding residues are identical for both the SWKB approximation and the quantum H-J formalism. The contour integral $J(E)$, matches exactly with the J_{SWKB}. This then explains the rather mysterious result of the exactness of SWKB approximation for the potentials in Table I.

We next consider the cases where lowest order SWKB fails to give exact answer. We at first look at the potential [12]

$$V(x) = \frac{1}{4}(x^2 + \frac{3}{4x^2}) - 2 + \frac{(3x^2 + x^4)}{(1 + x^4)} \quad , \tag{18}$$

with

$$\omega(x) = \frac{2x^6 + 3x^4 - x^2 - 6}{2x(x^2+1)(x^2+2)} \quad . \tag{19}$$

The SWKB quantization condition reads

$$J_{SWKB} = \frac{1}{2\pi}\oint_C \sqrt{E - \omega^2(x)}dx = n\hbar \quad , \tag{20}$$

It is noticed that the above integrand becomes zero at twelve points in the complex plane giving rise to six branch cuts. One of these branch cuts is included in the contour C. The integral around a branch cut on the negative axis is equal to the above integral due to the symmetry $x \to -x$.

The integrals around the other four branch cuts cannot be related to the integral J_C. We use J_{OBC} to denote the sum of the integrals along contours enclosing the four remaining branch cuts.

We now try to proceed, as far as possible, parallel to the Eckart potential. Let J_{Γ_R} be a circular contour with the center at origin and radius large enough to enclose all the singular points of the integrand. Note that the integrand has poles at $y = 0$; $y = \pm i$ and $y = \pm\sqrt{2}i$. The contour integral J_{Γ_R} is given by

$$J_{\Gamma_R} = 2n\hbar + J_{OBC} + J_{\gamma_1} + J_{\gamma_2} + J_{\gamma_3} + J_{\gamma_4} + J_{\gamma_5} \quad , \tag{21}$$

or

$$J_{\Gamma_R} - \sum_{p=1}^{5} J_{\gamma_p} = 2n\hbar - J_{OBC} \quad , \tag{22}$$

where $J_{\gamma_1}, J_{\gamma_2}, J_{\gamma_3}, J_{\gamma_4}$ and J_{γ_5} are the corresponding contour integrals for the contours $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ and γ_5, around $y = 0; +i; -i; +\sqrt{2}i$ and - $\sqrt{2}i$

Various contour integrals can be evaluated as before and we get $J_{\gamma_1} = 3/2$, $J_{\gamma_2} = -1$, $J_{\gamma_3} = -1$, $J_{\gamma_4} = 1$ and $J_{\gamma_5} = 1$. The signs of the residues needed have been fixed as for the Eckart

potential. J_{Γ_R} is again calculated making the mapping $y \to 1/z$ and looking for the residue of the pole at $z \to 0$. The corresponding contour integral is found to be,

$$J_{\Gamma_R} = 2E + 3/2 \quad . \tag{23}$$

The SWKB quantization condition is equivalent to Eq. (22). and this condition involves the unknown integral J_{OBC}. Therefore, this complex variables approach cannot be used to complete the calculation of energy eigenvalues in SWKB approximation for the potential (18). However, it is easily checked that if we substitute the values of $J_{\Gamma_R}, J_{\gamma_1}, J_{\gamma_2}, J_{\gamma_3}, J_{\gamma_4}$ and J_{γ_5} in

$$J_{\Gamma_R} - \sum_{p=1}^{5} J_{\gamma_p} = 2n\hbar \quad , \tag{24}$$

we obtain the exact eigenvalues.

Therefore, it is clear that if the sum of the contributions coming from other branch cuts is nonzero, the SWKB approximation will not be exact.

A similar analysis can be repeated for some of the other potentials. It is known that [12,5] for

$$V(x) = -1/x + \frac{x(x+2)}{(1+x+x^2/2)^2} + 1/16 \quad , \tag{25}$$

and

$$V(x) = (1-y^2)\left[-\lambda^2\nu(\nu+1) + \frac{1}{4}(1-\lambda^2)[2-(7-\lambda^2)y^2 + 5(1-\lambda^2)y^4]\right] \quad , \tag{26}$$

with super potentials,

$$\omega(x) = \frac{x^6 - 16x^4 - 56x^3 - 108x^2 - 240x - 192}{4x(x^2+2x+2)(x^3+6x^2+16x+24)} \quad , \tag{27}$$

and

$$\omega(x) = \frac{1}{2}(1-\lambda^2)y(y^2-1)\mu_0\lambda^2 y \quad , \tag{28}$$

SWKB does not give exact answers. We once again see that SWKB is not exact because of the contributions from the other branch cuts, that are present in both these cases.

V. CONCLUSION:

To conclude, we have shown using the complex contour integration technique, for several solvable potentials in one dimension, that SWKB gave exact results. We then compared it with the answers gotten using the quantum H-J method, from the *exact* quantization condition. This has provided some useful insight: It is the similarity of the singularity structure of $p(x,E)$ and $\sqrt{E-\omega^2}$ in the non-classical regions of the $x-$plane, namely the matching of the poles and the residues that is responsible for this exactness. Table I contains all the relevant details about the steps involved in the calculation.

For the non-exact potentials, it is the inability to deform the contour appropriately because of the presence of other poles and branch cuts in the complex $x-$plane that prevents an exact solution in SWKB approach.

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Table I : Hyperbolic and trigonometric potentials. The mapping used for hyperbolic potentials is $y = \exp(\alpha x)$, while for the trigonometric ones $y = \exp(i\alpha x)$ is being used; α is real and positive.

Name of potential	Super potential	Fixed poles at	αI_{γ_p}	Eigenvalue E_n
		0	$-i\sqrt{(E-A^2)}$	
Scarf II		i	$(iB-A)$	
(hyperbolic)	$A\tanh\alpha x + B\text{sech}\,\alpha x$	$-i$	$-(iB+A)$	$A^2-(A-\alpha\hbar)^2$
		∞	$i\sqrt{(E-A^2)}$	
Rosen -		0	$-i\sqrt{E-(A-\frac{B}{A})^2}$	A^2+B^2/A^2
Morse II	$A\tanh\alpha x + B/A$	i	$-A$	$-(A-\alpha\hbar)^2$
(hyperbolic)	$(B < A^2)$	$-i$	$-A$	$-B^2/(A-n\alpha\hbar)^2$
		∞	$i\sqrt{E-(A+\frac{B}{A})^2}$	
Generalised		0	$-i\sqrt{(E-A^2)}$	
Poschl-	$A\coth\alpha x - B\text{cosech}^2\,\alpha x$	1	$-(A-B)$	$A^2-(A-n\alpha\hbar)^2$
Teller	$(A < B$ and $x \geq 0)$	-1	$-(A+B)$	
(hyperbolic)		∞	$i\sqrt{(E-A^2)}$	
Scarf I		0	$\sqrt{(E+A^2)}$	
(trig-	$-A\tan\alpha x + B\sec\alpha x$	i	$-(A-B)$	$(A+n\alpha\hbar)^2-A^2$
onometric)	$(-\pi/2 \leq \alpha x \leq \pi/2)$	$-i$	$-(A+B)$	
		∞	$-\sqrt{(E+A^2)}$	
Rosen-		0	$-\sqrt{E+(A+i\frac{B}{A})^2}$	A^2+B^2/A^2
Morse-I	$-A\cot\alpha x - B/A$	1	A	$-(A+n\alpha\hbar)^2$
(trig-	$(0 \leq \alpha x \leq \pi)$	-1	A	$-B^2/(A+n\alpha\hbar)^2$
onometric)		∞	$\sqrt{E+(A-i\frac{B}{A})^2}$	

Figure Captions

1. Fig.1. Contour for the Eckart potential problem.